18.05 Lecture 4
February 11, 2005

Union of Events

$$\mathbb{P}(A_1 \cup ... \cup A_n) = \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i A_j) + \sum_{i<j<k} \mathbb{P}(A_i A_j A_k) + ...$$

It is often easier to calculate $\mathbb{P}$(intersections) than $\mathbb{P}$(unions)

Matching Problem: You have n letters and n envelopes, randomly stuff the letters into the envelopes. What is the probability that at least one letter will match its intended envelope?
$\mathbb{P}(A_1 \cup ... \cup A_n)$, $A_i = \{\text{ith position will match}\}$
$\mathbb{P}(A_i) = \frac{1}{n} = \frac{(n-1)!}{n!}$
(permute everyone else if just A_i is in the right place.)
$\mathbb{P}(A_i A_j) = \frac{(n-2)!}{n!}$ (A_i and A_j are in the right place)
$\mathbb{P}(A_{i1} A_{i2} ... A_{ik}) = \frac{(n-k)!}{n!}$

$$\mathbb{P}(A_1 \cup ... \cup A_n) = n \times \frac{1}{n} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} - ... + (-1)^{n+1}\binom{n}{n}\frac{(n-n)!}{n!}$$

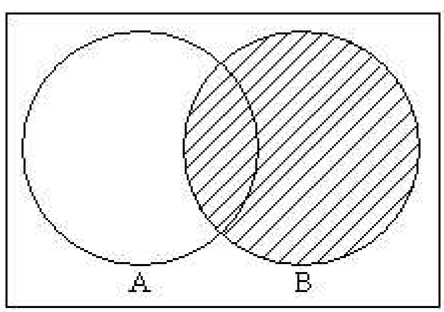
general term:

$$\binom{n}{k}\frac{(n-k)!}{n!} = \frac{n!(n-k)!}{k!(n-k)!n!} = \frac{1}{k!}$$

$$\text{SUM} = 1 - \frac{1}{2!} + \frac{1}{3!} - ... + (-1)^{n+1}\frac{1}{n!}$$

Recall: Taylor series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$
for x= -1, $e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + ...$
therefore, SUM = 1 - limit of Taylor series as n $\to \infty$
When n is large, the probability converges to $1 - e^{-1} = 0.63$

§2.1 - Conditional Probability
Given that B "happened," what is the probability that A also happened?
The sample space is narrowed down to the space where B has occurred:

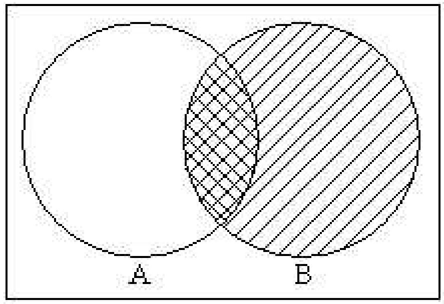


The sample size now only includes the determination that event B happened.

Definition: Conditional probability of Event A given Event B:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$$

Visually, conditional probability is the area shown below:



It is sometimes easier to calculate intersection given conditional probability:
$\mathbb{P}(AB) = \mathbb{P}(A|B)\mathbb{P}(B)$

Example: Roll 2 dice, sum (T) is odd. Find $\mathbb{P}(T < 8)$.
B = {T is odd}, A = {T < 8}

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}, \mathbb{P}(B) = \frac{18}{6^2} = \frac{1}{2}$$

All possible odd T = 3, 5, 7, 9, 11.
Ways to get T = 2, 4, 6, 4, 2 - respectively.
$\mathbb{P}(AB) = \frac{12}{36} = \frac{1}{3}; \mathbb{P}(A|B) = \frac{1/3}{1/2} = \frac{2}{3}$

Example: Roll 2 dice until sum of 7 or 8 results (T = 7 or 8)
$\mathbb{P}(A = \{T = 7\}), B = \{T = 7 \text{ or } 8\}$
This is the same case as if you roll once.
$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{6/36}{(6+5)/36} = \frac{6}{11}$

Example:

Treatments for a disease, results after 2 years:

Result	A	B	C	Placebo
Relapse	18	13	22	24
No Relapse	22	25	16	10

Example, considering Placebo: B = Placebo, A = Relapse. $\mathbb{P}(A|B) = \frac{24}{24+10} = 0.7$
Example, considering treatment B: $\mathbb{P}(A|B) = \frac{13}{13+25} = 0.34$

As stated earlier, conditional probability can be used to calculate intersections:
Example: You have r red balls and b black balls in a bin.
Draw 2 without replacement, What is $\mathbb{P}(1 = \text{red}, 2 = \text{black})$?
What is $\mathbb{P}(2 = \text{black})$ given that 1 = red ? $\mathbb{P}(1 = \text{red}) = \frac{r}{r+b}$
Now, there are only r - 1 red balls and still b black balls.
$\mathbb{P}(2 = \text{black}|1 = \text{red}) = \frac{b}{b+r-1} \rightsquigarrow \mathbb{P}(AB) = \frac{b}{b+r-1} \times \frac{r}{r+b}$

$$\mathbb{P}(A_1A_2...A_n) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_2|A_1) \times ... \times \mathbb{P}(A_n|A_{n-1}...A_2|A_1) =$$

$$= \mathbb{P}(A_1) \times \frac{\mathbb{P}(A_2A_1)}{\mathbb{P}(A_1)} \times \frac{\mathbb{P}(A_3A_2A_1)}{\mathbb{P}(A_2A_1)} \times ... \frac{\mathbb{P}(A_nA_{n-1}...A_1)}{\mathbb{P}(A_{n-1}...A_1)} =$$

$$= \mathbb{P}(A_nA_{n-1}...A_1)$$

Example, continued: Now, find $\mathbb{P}(r, b, b, r)$

$$= \frac{r}{r+b} \times \frac{b}{r-1+b} \times \frac{b-1}{r+b-2} \times \frac{r-1}{r+b-3}$$

Example, Casino game - Craps. What's the probability of actually winning??
On first roll: 7, 11 - win; 2, 3, 12 - lose; any other number (x_1), you continue playing.
If you eventually roll 7 - lose; x_1, you win!

$$\mathbb{P}(win) = \mathbb{P}(x_1 = 7 \text{ or } 11) + \mathbb{P}(x_1 = 4)\mathbb{P}(\text{get 4 before } 7 | x_1 = 4) +$$

$$+\mathbb{P}(x_1 = 5)\mathbb{P}(\text{get 5 before } 7 | x_1 = 5) + ... = 0.493$$

The game is almost fair!

** End of Lecture 4